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SELF-EQUILIBRATED STRESS FIELDS IN A CONTINUOUS MEDIUM

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It is proved that the solutions of the static equations of a continuous medium constructed in terms of a stress function are self-equilibrated. From a mathematical point of view, these functions can be treated as the connectivity coefficients of the intrinsic geometry of the medium. It is shown that from a physical point of view, the existence of self-equilibrated stress fields is due to a nonuniform entropy distribution in the medium. As an example, for a circle in polar coordinates and a cylindrical sample, a self-equilibrated stress field and an elastic field compensating for its surface component are constructed and it is shown how to write the equation for the intrinsic geometrical characteristics.

Key words: *self-equilibrated stress fields, stress function, entropy.*

Introduction. In the mechanics of continuous media, it is well known that for specified motion of a continuous medium and specified distributions of surface and mass forces, the stress field in the medium is not uniquely determined [1]. This result is based on the fact that the equations of mechanical equilibrium for a continuous medium in the absence of external forces

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \qquad \sigma_{ij} n_j \Big|_{\partial V} = 0 \tag{1}$$

can have nontrivial solutions. In (1), σ_{ij} are the stress-tensor components and n_i are the coordinates of the normal vector to the boundary; summation is performed over repeating subscripts. The nontrivial solution of system (1) constructed in [1] for a unit cube has the form

$$\begin{gathered}\sigma_{11} = \cos \pi x_1 \cos \pi x_2 + \cos \pi x_2, \qquad \sigma_{22} = \cos \pi x_1 \cos \pi x_2 + \cos \pi x_1,\\ \sigma_{12} = \sigma_{21} = \sin \pi x_1 \sin \pi x_2, \qquad \sigma_{33} = \sigma_{13} = \sigma_{31} = \sigma_{32} = \sigma_{23} = 0.\end{gathered} \tag{2}$$

Nevertheless, the nonuniqueness of the stress field turns out to be useful for correcting the equations of state of continuous media. In particular, such correction allowed Godunov and Romenskii [1, § 31], without changing the smooth solutions of elasticity theory, to change the equation of state of the nonlinear theory of elasticity so that it became strictly convex and to bring the system of conservation laws to a symmetric hyperbolic form. We note that the equation of state corresponding to solution (2) is not strictly convex. Indeed, let σ_{ij} be expressed in terms of the strain tensor ε_{ij} according to Murnaghan's formulas [1]:

$$\sigma_{ij} = \partial W / \partial \varepsilon_{ij}. \tag{3}$$

We introduce the strain tensor ε_{ij} and the elastic potential W in the form

$$\begin{gathered}\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \qquad u_i = \frac{\partial \Phi}{\partial x_i},\\ \Phi = -(\cos \pi x_1 \cos \pi x_2 + \cos \pi x_1 + \cos \pi x_2)/\pi^2;\\ W = (\varepsilon_{11} + \varepsilon_{22})^2/2 - (\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\varepsilon_{12}^2) = \varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2.\end{gathered} \tag{4}$$

*Deceased.

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Substituting (4) into (3), we obtain (2). In this case, W is not a strictly convex function ε_{ij} since the matrix of the second derivatives $\partial^2 W/\partial\varepsilon_{ij}\,\partial\varepsilon_{pq}$ is not strictly positive. Interest in the nontrivial solutions of system (1) is motivated by the necessity of solving technological problems. The fact is that technologists are well aware of the existence of residual stresses in articles from various materials [2]. An example are welds. From the experimental data of [2] it follows that weld stresses can have values comparable to stresses induced by external action. The bodies in this case are in mechanical and thermal equilibrium. The existence of residual stresses implies that in the equilibrium equations (1), the stresses σ_{ij} inside the body are not equal to zero and for an arbitrary element of volume ω inside the body, the following integral equilibrium conditions are satisfied:

$$\int\limits_{\partial\omega} X_i\,dS = \int\limits_{\partial\omega} \sigma_{ij} n_j\,dS = 0 \qquad (i,j = 1,2,3); \tag{5}$$

$$M_{ij} = \int\limits_{\partial\omega} (\sigma_{ik}x_j - \sigma_{jk}x_i) n_k\,dS + \int\limits_{\omega} (\sigma_{ji} - \sigma_{ij})\,dV = 0. \tag{6}$$

The nontrivial solutions of system (1) that satisfy relations (5) and (6) are called self-equilibrated. In the present paper, we prove that the nontrivial solutions of system (1) constructed in a previous study [3] are self-equilibrated. Solution (2) is an example of such a self-equilibrated stress field. For the case of small strains of a continuous medium and an isotropic intrinsic metric, it is shown that the self-equilibrated stresses are determined by a nonuniform entropy distribution in the material. For a circle in polar coordinates and a cylindrical sample, possible versions of constructing equations for the intrinsic metric are given.

Self-Equilibration Property. In [3], the nontrivial solutions of Eqs. (1) are written in terms of the stress function $\Gamma_{qm,p}$ as

$$\sigma_{ij} = 2\sigma_0 l^2 \varepsilon_{ipq}\varepsilon_{jmk}\,\frac{\partial\Gamma_{qm,p}}{\partial x_k}, \tag{7}$$

where ε_{ipq} is the Levi-Civita symbol. The constants σ_0 and l have the dimensions of stress and length, respectively. Expressions (7) satisfy Eqs. (1) identically.

We check that conditions (5) and (6) are satisfied for a field σ_{ij}. The validity of condition (5) is obvious since after substitution of (7) into (5), the obtained integrand coincides with the normal component of the curl of the tensor field, and the closed-surface integral of the component is known to be equal to zero:

$$\int\limits_{\partial\omega} dS\,\sigma_{ij} n_j = -2\sigma_0 l^2 \int\limits_{\partial\omega} dS\,n_j \varepsilon_{jkm}\,\frac{\partial}{\partial x_k}\,\Gamma_{qm,p}\varepsilon_{ipq} = 0.$$

Generally, solution (7) does not guarantee symmetry of the stress tensor, and the proof of condition (6) is a separate problem. Using (7), we convert the volume integral in (6) to the following integral over the boundary:

$$\int\limits_{\omega} dV\,(\sigma_{ji} - \sigma_{ij}) = 2\sigma_0 l^2 \int\limits_{\partial\omega} dS\,n_k(\varepsilon_{jpq}\varepsilon_{imk} - \varepsilon_{ipq}\varepsilon_{jmk})\Gamma_{qm,p}. \tag{8}$$

Since

$$\int\limits_{\partial\omega} dS\,\sigma_{ik} n_k x_j = 2\sigma_0 l^2 \int\limits_{\partial\omega} dS\,n_k \varepsilon_{ipq}\varepsilon_{kml}\,\frac{\partial}{\partial x_l}\,(\Gamma_{qm,p}x_j) - 2\sigma_0 l^2 \int\limits_{\partial\omega} dS\,n_k \varepsilon_{ipq}\varepsilon_{kmj}\Gamma_{qm,p},$$

the surface integral in (6) can be written as

$$\int\limits_{\partial\omega} dS\,(\sigma_{ik}x_j - \sigma_{jk}x_i) n_k = 2\sigma_0 l^2 \int\limits_{\partial\omega} \varepsilon_{ipq} dS\,n_k\Big[\varepsilon_{kml}\,\frac{\partial}{\partial x_l}\,(\Gamma_{qm,p}x_j) - \varepsilon_{jpq}\varepsilon_{kml}\,\frac{\partial}{\partial x_l}\,(\Gamma_{qm,p}x_i)\Big]$$

$$+ 2\sigma_0 l^2 \int\limits_{\partial\omega} dS\,n_k(\varepsilon_{ipq}\varepsilon_{jmk} - \varepsilon_{jpq}\varepsilon_{imk})\Gamma_{qm,p}. \tag{9}$$

Equations (8) and (9) imply

$$M_{ij} = \int\limits_{\partial\omega} (\sigma_{ik}x_j - \sigma_{jk}x_i) n_k\,dS + \int\limits_{\omega} (\sigma_{ji} - \sigma_{ij})\,dV = 2\sigma_0 l^2 \int\limits_{\partial\omega} dS\,n_k \varepsilon_{kml}\,\frac{\partial}{\partial x_l}\Big[\varepsilon_{ipq}(\Gamma_{qm,p}x_j) - \varepsilon_{jpq}(\Gamma_{qm,p}x_i)\Big].$$

The integrand in the surface integral coincides with the normal component of the curl of the tensor field and the closed-surface integral of this component is equal to zero.

Generally, the pointwise boundary conditions (1) used in elasticity theory are not satisfied for the stress field (7): $\sigma_{ij}n_j\Big|_{\partial V} \neq 0$. By virtue of the linearity of the equilibrium equations, it was proposed [3] to introduce an elastic stress field $\pi_{ij} = \pi_{ji}$, so that

$$\frac{\partial \pi_{ij}}{\partial x_j} = 0, \qquad \pi_{ij}n_j\Big|_{\partial V} = -\sigma_{ij}n_j\Big|_{\partial V}.$$

The stress field

$$T_{ij} = \pi_{ij} + \sigma_{ij} \tag{10}$$

satisfies the equilibrium equations (1) and satisfies the condition of absence of external forces on the field surface in a pointwise sense:

$$\frac{\partial T_{ij}}{\partial x_j} = 0, \qquad T_{ij}n_j\Big|_{\partial V} = 0. \tag{11}$$

Thus, in the equilibrium state, the stress field T_{ij} can be written as the sum of the self-equilibrated σ_{ij} and elastic π_{ij} fields.

Relationship of the Self-Equilibrated Solution with the Classical Theory of Elasticity. In physical theories of strength and plasticity, nonzero stresses in equilibrium occur in models that take into account crystal structure defects in materials (see, for example, [4, Sec. IV]). In the 1950s, an analysis of such physical models led Kondo [5] and Bilby [6] to the conclusion that stresses must be described using geometrical objects forbidden by the classical theory of elasticity. Godunov, analyzing the necessity of generalizing classical theory, pointed out that the identification of changes in the intrinsic metric of a material g_{ij}, which governs changes in its internal energy, with the corresponding body shape changes in the Euclidean metric of an external observer is an additional hypothesis postulated in classical theory.

In view of the necessity of introducing non-Euclidean objects to describe defects, it is proposed [3] to treat $\Gamma_{qm,p}$ in (7) as connectivity objects on the manifold generated by the internal defect structure of the material. The reserve of the functions $\Gamma_{qm,p}$ is large enough, and a decrease in it is related to the hypothesis on the geometrical structure of the manifold considered. In particular, if the manifold is Riemannian, then $\Gamma_{qm,p}$ are expressed in terms of the metric under by the Christoffel formulas [7]

$$\Gamma_{ij,k} = \frac{1}{2}\Big(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\Big).$$

Then the self-equilibrated stresses (7) are completely defined by the metric of the manifold:

$$\sigma_{ij} = 2\sigma_0 l^2 \varepsilon_{ipq}\varepsilon_{jmn}\frac{\partial^2 g_{pm}}{\partial x_n \partial x_q}. \tag{12}$$

Let the intrinsic metric be diagonal $g_{ij} = \delta_{ij}g$. Using the identity

$$\varepsilon_{ipq}\varepsilon_{jpn} = \delta_{ij}\delta_{qn} - \delta_{in}\delta_{qj},$$

we can write (12) as

$$\sigma_{ij} = 2\sigma_0 l^2\Big(\delta_{ij}\,\Delta g - \frac{\partial^2 g}{\partial x_i \partial x_j}\Big) \tag{13}$$

(Δ is the Laplacian). From (3) and (4), it follows that σ_{ij} in (2) is similar in structure to (13):

$$\sigma_{ij} = \delta_{ij}\,\Delta\Phi - \frac{\partial^2 \Phi}{\partial x_i \partial x_j}$$

($\Phi = 2\sigma_0 l^2 g$). Consequently, solution (2) constructed in [1] has the self-equilibration property, and its intrinsic metric is diagonal.

Let us consider small strains of a continuous medium for an isothermal process. In this case, the elastic stress tensor π_{ij} is related to the elastic strain tensor ε_{ij} by Hooke's law:

$$\pi_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}. \tag{14}$$

Here λ and μ are Lamé's parameters. The total field T_{ij} is defined by relation (10), which, with allowance for (13) and (14), is written as

$$T_{ij} = \delta_{ij}(\lambda\varepsilon_{kk} + 2\sigma_0 l^2 \Delta g) + 2\mu\varepsilon_{ij} - 2\sigma_0 l^2 \frac{\partial^2 g}{\partial x_i \partial x_j}. \tag{15}$$

We rearrange relation (15). For this, we write

$$2\mu\varepsilon_{ij} - 2\sigma_0 l^2 \frac{\partial^2 g}{\partial x_i \partial x_j} = 2\mu\Big[\frac{1}{2}\frac{\partial}{\partial x_j}\Big(u_i - \frac{\sigma_0 l^2}{\mu}\frac{\partial g}{\partial x_i}\Big) + \frac{1}{2}\frac{\partial}{\partial x_i}\Big(u_j - \frac{\sigma_0 l^2}{\mu}\frac{\partial g}{\partial x_j}\Big)\Big].$$

Introducing the functions

$$U_i = u_i - \frac{\sigma_0 l^2}{\mu}\frac{\partial g}{\partial x_i}, \tag{16}$$

we have

$$2\mu\varepsilon_{ij} - 2\sigma_0 l^2 \frac{\partial^2 g}{\partial x_i \partial x_j} = 2\mu \frac{1}{2}\Big(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\Big) = 2\mu E_{ij}. \tag{17}$$

Substituting relations (16) and (17) into (15), we obtain

$$T_{ij} = \delta_{ij}\Big[\lambda \frac{\partial U_k}{\partial x_k} + \sigma_0 l^2 \frac{\lambda + 2\mu}{\mu}\Delta g\Big] + 2\mu E_{ij}.$$

We introduce the first invariant E_1 of the tensor E_{ij}, the scalar function $s_c - s_0$, and the phenomenological parameter η, assuming that

$$E_1 = \frac{\partial U_k}{\partial x_k}, \qquad \sigma_0 l^2 \frac{\lambda + 2\mu}{\mu}\Delta g = -\eta(s_c - s_0). \tag{18}$$

This allows expression (15) for T_{ij} to be written as

$$T_{ij} = \delta_{ij}[\lambda E_1 - \eta(s_c - s_0)] + 2\mu E_{ij}. \tag{19}$$

Formula (19) suggests a relationship with classical theory and reveals the physical meaning of the intrinsic metric. Since we consider small strains of a continuous medium, in the internal energy function U, we restrict ourselves to the first and second terms of the expansion in powers of the strain tensor E_{ij} and the deviation of the entropy s from a certain fixed value s_0. With accuracy up to an additive constant, the energy is written as [1, p. 63]

$$\rho_0 U = \rho_0 T(s - s_0) + \lambda(E_{kk})^2/2 + \mu E_{ij}E_{ij} - \eta(s - s_0)E_{kk} + \xi\rho_0(s - s_0)^2/2, \tag{20}$$

where ρ_0 is the density of the medium in the initial state, T is the temperature, and ξ is a phenomenological parameter. The equation of state is determined by Murnaghan's relations [1]

$$T_{ij} = \rho_0 \frac{\partial U}{\partial E_{ij}}. \tag{21}$$

Substituting (20) into (21), we obtain the stress tensor

$$T_{ij} = \delta_{ij}[\lambda E_{kk} - \eta(s - s_0)] + 2\mu E_{ij}. \tag{22}$$

From (19) and (22) it follows that the scalar function introduced above s_c coincides with the entropy of the medium $s = s_c$. The meaning of this result is as follows. We consider an ideal crystal in an equilibrium state. From a physical point of view, an ideal crystal is a lattice whose sites are occupied by atoms; the lattice has a particular symmetry group and there are no defects in an ideal crystal. In an isolated system, defects can form, for example, when atoms lose unstable equilibrium as a result of a fluctuation. Such a transition leads to a change in the configuration of the crystal lattice, its rearrangement, and formation of a more stable state of equilibrium, resulting in an increase in the entropy. The new initial state differs from the original by the vector ∇g (16); in this case, g is the solution of Poisson's equation (18), whose right side is determined by the configurational contribution s_c into the entropy.

Let the entropy be fixed, i.e., $s_c = s_0$. Then,

$$\Delta g = 0. \tag{23}$$

From (16) and (23), we obtain $\operatorname{div} \boldsymbol{U} = \operatorname{div} \boldsymbol{u}$. The quantities $\operatorname{div} \boldsymbol{U}$ and $\operatorname{div} \boldsymbol{u}$ are the first invariants of the tensors

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \qquad E_{ij} = \frac{1}{2}\Big(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\Big),$$

respectively. The invariant quantity $\varepsilon_{kk} = E_{kk}$ defines (in a linear approximation) the density of the medium $\rho = \rho_0(1 - \varepsilon_{kk})$. In physics, gradient transformations similar to (16) arise, for example, in determining the electromagnetic field in terms of potentials in field theory [8]. In this case, the electromagnetic field is invariant under gradient (invariance) transformations and the potentials are not uniquely determined. Because of the nonunique determination of the potentials, they can be chosen so that they satisfy one arbitrary condition — the calibration condition. By analogy with field theory, one can say that condition (23) fixes the choice of the function g (i.e., the choice of calibration) with introduction of the displacement vector for points of the continuous medium and the invariant quantity is the density of the medium ρ.

Self-Equilibrated Stress Field in a Circle. We assume that the intrinsic metric g_{ij} is diagonal: $g_{ij} = \delta_{ij} g$. Then, in polar coordinates, the components of the self-equilibrated field σ_{ij} (13) (with accuracy up to the normalization factor) are equal to [9]

$$\sigma_{rr} = \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \varphi^2}, \qquad \sigma_{\varphi\varphi} = \frac{\partial^2 g}{\partial r^2}, \qquad \sigma_{r\varphi} = -\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial g}{\partial \varphi}\Big). \tag{24}$$

The description of the structure of the stress field remains unclosed because a method for constructing the function g is not indicated.

Let us find out in what form it is possible to write the equation for g. As shown above, g is a characteristic of the internal structure of the material and is determined by the configurational entropy $s_c - s_0$ [see (18)]. For a uniform entropy distribution, i.e., for $s_c = s_0$, Eq. (23) is satisfied. A simple modification of this equation for $s_c \neq s_0$ arises from perturbation of its right side:

$$\Delta g = -k^2 g. \tag{25}$$

The quantity k is a parameter of the model.

In the polar coordinates, the solution of Eq. (25) is given by the formula $g = J_n(kr) \cos n\varphi$, where J_n is a cylindrical function of the first kind. From this and from (24), we obtain the components of the self-equilibrated stress field:

$$\begin{gathered} \sigma_{rr} = \frac{1}{r^2}\Big(r\frac{dJ_n}{dr} - n^2 J_n\Big)\cos n\varphi, \qquad \sigma_{r\varphi} = n\frac{d}{dr}\Big(\frac{J_n}{r}\Big)\sin n\varphi, \\ \sigma_{\varphi\varphi} = -\frac{1}{r^2}\Big(r\frac{dJ_n}{dr} - n^2 J_n + k^2 r^2 J_n\Big)\cos n\varphi. \end{gathered} \tag{26}$$

We consider the case $n \geqslant 2$. In the polar coordinates, the elastic field satisfies the equations

$$\frac{\partial \pi_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \pi_{r\varphi}}{\partial \varphi} + \frac{\pi_{rr} - \pi_{\varphi\varphi}}{r} = 0, \qquad \frac{\partial \pi_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial \pi_{\varphi\varphi}}{\partial \varphi} + \frac{2\pi_{r\varphi}}{r} = 0. \tag{27}$$

We first construct a solution of Eqs. (27), and then, using this field, calculate the corresponding displacements. We write a particular solution of system (27) in the form

$$\pi_{rr} = Ar^{n-2}\cos n\varphi, \qquad \pi_{\varphi\varphi} = -Ar^{n-2}\cos n\varphi, \qquad \pi_{r\varphi} = -Ar^{n-2}\sin n\varphi. \tag{28}$$

The components of the elastic strain tensor ε_{ij} are related to the stresses (28) by Hooke's law (14). From (28) it follows that $\pi_{rr} + \pi_{\varphi\varphi} = 0$; then, the first invariant of the elastic strain tensor is equal to zero: $\varepsilon = (\pi_{rr} + \pi_{\varphi\varphi})/(\lambda + 2\mu) = 0$.

Let us calculate the displacement fields u_r and u_φ:

$$u_r = \frac{A}{2\mu\lambda(n-1)}\, r^{n-1}\cos n\varphi, \qquad u_\varphi = -\frac{A}{2\mu\lambda(n-1)}\, r^{n-1}\sin n\varphi. \tag{29}$$

The total stresses (10) are defined by the formulas

$$T_{rr} = \sigma_{rr} + \pi_{rr}, \qquad T_{\varphi\varphi} = \sigma_{\varphi\varphi} + \pi_{\varphi\varphi}, \qquad T_{r\varphi} = \sigma_{r\varphi} + \pi_{r\varphi}.$$

The boundary conditions (11) lead to the following equations:

$$\frac{1}{r^2}\Big(r\,\frac{dJ_n}{dr} - n^2 J_n\Big) + Ar^{n-2}\Big|_{r=R} = 0, \qquad \frac{n}{r^2}\Big(r\,\frac{dJ_n}{dr} - J_n\Big) - Ar^{n-2}\Big|_{r=R} = 0.$$

Eliminating the coefficient A, we have

$$\frac{dJ_n}{dr} - \frac{n}{r}\,J_n\Big|_{r=R} = 0. \tag{30}$$

Using the recursion formulas for cylindrical functions [10], we rewrite Eq. (30) as

$$J_{n+1}\Big|_{r=R} = 0. \tag{31}$$

For $n = 1$, formulas (29) have a singularity. It should be noted that after substitution of (26) for $n = 1$ into the boundary conditions (11), we obtain

$$r\,\frac{dJ_1}{dr} - J_1\Big|_{r=R} = 0, \qquad \frac{d}{dr}\Big(\frac{J_1}{r}\Big)\Big|_{r=R} = 0. \tag{32}$$

Obviously, for $r \neq 0$, the conditions in (32) are equivalent. Then, using the recursion formulas for cylindrical functions [10], we write (32) as

$$J_2\Big|_{r=R} = 0. \tag{33}$$

Relation (33) is a special case of (31) for $n = 1$. Thus, for $n = 1$, the field (26) satisfies the boundary conditions (11) in a pointwise sense; therefore, the elastic field is equal to zero. However, as noted above, the equation of state corresponding to this solution is not strictly convex.

Self-Equilibrated Stress Field for a Cylindrical Sample. We consider a cylindrical sample which is in equilibrium. By analogy with the plane case, we assume that the first invariant of the strain tensor is equal to zero. The equilibrium equations are written in displacements as follows [11]:

$$\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\,\frac{\partial u_\varphi}{\partial \varphi} = 0, \qquad \Delta u_\varphi - \frac{u_\varphi}{r^2} + \frac{2}{r^2}\,\frac{\partial u_r}{\partial \varphi} = 0, \qquad \Delta u_z = 0. \tag{34}$$

The solution for the displacement vector components is sought in the form

$$u_r = u(r)\cos n\varphi \cos\gamma z, \qquad u_\varphi = v(r)\sin n\varphi\cos\gamma z, \qquad u_z = w(r)\cos n\varphi\sin\gamma z, \tag{35}$$

where $u(r)$, $v(r)$, and $w(r)$ are unknown functions, n is an integer, and the parameter γ is defined below. Substituting (35) into (34), we obtain the following equations for $u = u(r)$, $v = v(r)$, and $w = w(r)$:

$$Lu - \frac{u}{x^2} - \frac{2n}{x^2}\,v = 0, \qquad Lv - \frac{v}{x^2} - \frac{2n}{x^2}\,u = 0, \qquad Lw = 0,$$

$$L = \frac{d^2}{dx^2} + \frac{1}{x}\,\frac{d}{dx} - \frac{n^2}{x^2} - 1, \qquad x = \gamma r.$$

The solution of this system is written as

$$u = \frac{A_1}{2}\,I_{n+1}(x) + \frac{A_2}{2}\,I_{n-1}(x), \qquad v = \frac{A_1}{2}\,I_{n+1}(x) - \frac{A_2}{2}\,I_{n-1}(x), \qquad w = A_3 I_n(x),$$

where $I_n(x)$, $I_{n+1}(x)$, and $I_{n-1}(x)$ are real cylindrical functions of an imaginary argument. The vanishing condition for the first strain invariant in terms of displacements yields constraints on the constants A_1, A_2, and A_3 in the form $2A_3 + A_1 + A_2 = 0$. The reference formulas relating the elastic stress tensor components to the displacement vector components u_r, u_φ, and u_z are well known [11]; therefore, we shall give the final result for the elastic stress field π_{ij}:

$$\pi_{rr} = \mu\gamma\Big[A_1\Big(I_n\Big(1 + \frac{2n^2 + 2n}{x^2}\Big) - \frac{n+1}{x} I_{n-1}\Big) + A_2\Big(I_n + \frac{n-1}{x}\,I_{n-1}\Big)\Big]\cos n\varphi\cos\gamma z,$$

$$\pi_{\varphi\varphi} = \mu\gamma\Big[A_1\Big(\frac{n+1}{x}\,I_{n-1} - \frac{2n^2+2n}{x^2}\,I_n\Big) - A_2\,\frac{n-1}{x}\,I_{n-1}\Big]\cos n\varphi\cos\gamma z,$$

$$\pi_{zz} = -\mu\gamma(A_1+A_2)I_n\cos n\varphi\cos\gamma z, \tag{36}$$

$$\pi_{\varphi r} = \frac{\mu\gamma}{2}\,A_1\Big[I_n\Big(1+\frac{4n^2+4n}{x^2}\Big) - \frac{2n+2}{x}\,I_{n-1}\Big]\sin n\varphi\cos\gamma z$$

$$-\frac{\mu\gamma}{2}\,A_2\Big(I_n + \frac{2n-2}{x}\,I_{n-1}\Big)\sin n\varphi\cos\gamma z,$$

$$\pi_{rz} = -\frac{\mu\gamma}{2}\Big[A_1\Big(2I_{n-1} - \frac{3n}{x}\,I_n\Big) + A_2\Big(2I_{n-1} - \frac{n}{x}\,I_n\Big)\Big]\cos n\varphi\sin\gamma z,$$

$$\pi_{\varphi z} = -\frac{\mu\gamma}{2}\Big[A_1\Big(I_{n-1} - \frac{3n}{x}\,I_n\Big) - A_2\Big(I_{n-1} + \frac{n}{x}\,I_n\Big)\Big]\sin n\varphi\sin\gamma z.$$

The self-equilibrated stress field is written as

$$\sigma_{rr} = f_{rr}(x)\cos n\varphi\cos\gamma z, \qquad \sigma_{r\varphi} = f_{r\varphi}(x)\sin n\varphi\cos\gamma z,$$

$$\sigma_{\varphi\varphi} = f_{\varphi\varphi}(x)\cos n\varphi\cos\gamma z, \qquad \sigma_{\varphi z} = f_{\varphi z}(x)\sin n\varphi\sin\gamma z, \tag{37}$$

$$\sigma_{zz} = f_{zz}(x)\cos n\varphi\cos\gamma z, \qquad \sigma_{rz} = f_{rz}(x)\cos n\varphi\sin\gamma z.$$

At the same time, the components of the self-equilibrated stress field are calculated in terms of the function g by formulas (13), which, in cylindrical coordinates have the following form (with accuracy up to the normalization factor):

$$\sigma_{rr} = \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial\varphi^2} + \frac{\partial^2 g}{\partial z^2}, \qquad \sigma_{\varphi\varphi} = \frac{\partial^2 g}{\partial r^2} + \frac{\partial^2 g}{\partial z^2}, \qquad \sigma_{zz} = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial\varphi^2},$$

$$\sigma_{r\varphi} = -\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial g}{\partial\varphi}\Big), \qquad \sigma_{\varphi z} = -\frac{1}{r}\frac{\partial^2 g}{\partial\varphi\,\partial z}, \qquad \sigma_{rz} = -\frac{\partial^2 g}{\partial r\,\partial z}. \tag{38}$$

Let the function g depend on the coordinates φ and z as follows:

$$g = G(x)\cos n\varphi\cos\gamma z. \tag{39}$$

Substituting (39) into (38) and using (37), we obtain the following representation for the functions $f_{ij} = f_{ij}(x)$ in terms of the function $G = G(x)$:

$$f_{rr} = \gamma^2\Big(\frac{1}{x}\frac{dG}{dx} - \frac{n^2}{x^2}\,G - G\Big), \qquad f_{r\varphi} = \gamma^2 n\,\frac{d}{dx}\Big(\frac{G}{x}\Big), \qquad f_{rz} = \gamma^2\,\frac{dG}{dx},$$

$$f_{\varphi z} = -\gamma^2\,\frac{n}{x}\,G, \qquad f_{\varphi\varphi} = \gamma^2\Big(\frac{d^2G}{dx^2} - G\Big), \qquad f_{zz} = \gamma^2\Big(\frac{d^2G}{dx^2} + \frac{1}{x}\frac{dG}{dx} - \frac{n^2}{x^2}\,G\Big). \tag{40}$$

Let us consider the boundary conditions for the components of the total stress field T_{ij}. We write the parameter γ as $\gamma = m\pi/h$, where m is a natural number. Then, the boundary conditions for the components $T_{rz} = \sigma_{rz} + \pi_{rz}$ and $T_{\varphi z} = \sigma_{\varphi z} + \pi_{\varphi z}$ for $z = \pm h$ are satisfied identically. The boundary condition for $T_{zz}|_{z=\pm h} = \sigma_{zz} + \pi_{zz}|_{z=\pm h} = 0$ leads to the following equation for the function G:

$$\frac{d^2G}{dx^2} + \frac{1}{x}\frac{dG}{dx} - \frac{n^2}{x^2}\,G = \frac{\mu}{\gamma}\,(A_1+A_2)I_n(x). \tag{41}$$

The solution of Eq. (41) is found by the formula

$$G(x) = Cx^n + \mu(A_1+A_2)I_n/\gamma. \tag{42}$$

Substituting (42) into (40) and using (37) to calculate the components of the field of self-equilibrated stresses, we obtain

$$\sigma_{rr} = C\gamma^2 x^n\Big(\frac{n-n^2}{x^2} - 1\Big) + \mu\gamma(A_1 + A_2)\Big[I_{n-1}\,\frac{1}{x} - I_n\Big(1 + \frac{n^2+n}{x^2}\Big)\Big]\cos n\varphi\cos\gamma z,$$

$$\sigma_{\varphi\varphi} = C\gamma^2 x^n\Big(\frac{n^2-n}{x^2} - 1\Big) + \mu\gamma(A_1 + A_2)\Big(I_n\,\frac{n^2+n}{x^2} - \frac{1}{x}\,I_{n-1}\Big)\cos n\varphi\cos\gamma z,$$

$$\sigma_{zz} = \mu\gamma(A_1 + A_2)I_n\cos n\varphi\cos\gamma z, \tag{43}$$

$$\sigma_{\varphi r} = C\gamma^2 n(n-1)x^{n-2} + \mu\gamma(A_1 + A_2)\Big(I_{n-1}\,\frac{n}{x} - I_n\,\frac{n^2+n}{x^2}\Big)\sin n\varphi\cos\gamma z,$$

$$\sigma_{rz} = C\gamma^2 n x^{n-1} + \mu\gamma(A_1 + A_2)\Big(I_{n-1} - I_n\,\frac{n}{x}\Big)\cos n\varphi\sin\gamma z,$$

$$\sigma_{\varphi z} = -C\gamma^2 n x^{n-1} + \mu\gamma(A_1 + A_2)\,\frac{n}{x}\,I_n\sin n\varphi\sin\gamma z.$$

Let us consider the boundary conditions on the surface of the cylinder

$$\sigma_{rr} + \pi_{rr}\Big|_{x=\gamma R} = 0, \qquad \sigma_{r\varphi} + \pi_{r\varphi}\Big|_{x=\gamma R} = 0, \qquad \sigma_{rz} + \pi_{rz}\Big|_{x=\gamma R} = 0.$$

Substituting the expressions for the stress components from (43) and (36) into the above formulas, we obtain the following system of linear algebraic equations for A_1, A_2, and C:

$$C\gamma x^n((n-n^2)/x^2 - 1)/\mu + A_1\Psi(x) + A_2\Theta(x)\Big|_{x=\gamma R} = 0,$$

$$2C\gamma x^{n-2}n(n-1)/\mu + A_1\Psi_1(x) + A_2\Theta_1(x)\Big|_{x=\gamma R} = 0, \tag{44}$$

$$2C\gamma x^{n-1}n/\mu + A_1\Psi_2(x) + A_2\Theta_2(x)\Big|_{x=\gamma R} = 0.$$

Here

$$\Psi(x) = -\Theta(x) = -I_{n-1}n/x + I_n n(n+1)/x^2,$$

$$\Psi_1(x) = -\Theta_1(x) = -2I_{n-1}/x + I_n(1 + (2n^2 + 2n)/x^2),$$

$$\Psi_2(x) = -\Theta_2(x) = I_n n/x.$$

A nontrivial solution of system (44) exists provided that the determinant of the system is equal to zero. This condition is satisfied since in the determinant, the second and third columns differ only in sign.

Therefore, in constructing the stress field for a cylindrical sample, it is necessary to specify the parameters C, A_1, and γ (or m). Values of these parameters depend on the conditions of material processing, and to determine them, one needs to perform additional experiments [2].

Conclusions. The internal stresses field in a material is composed of a self-equilibrated stress field and an elastic stress field, which compensates for the surface disequilibrium of the self-equilibrated stresses. The combined effect of these stresses allows the sample to retain the specified shape. From a physical point of view, the existence of nonzero self-equilibrated stress fields in a continuous medium is due to the presence of structural defects in the material, whose description requires introducing an intrinsic metric tensor. Therefore, an analysis of dissipative processes in materials based on non-Euclidean geometrical models considered previously [12–14] will allow one to solve the problem of the structure of self-equilibrated stress fields in materials under processing treatment.

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REFERENCES

1. S. K. Godunov and E. I. Romenskii, *Elements of the Mechanics of Continuous Media* [in Russian], Nauchnaya Kniga, Novosibirsk (1998).
2. G. N. Chernyshev, A. L. Popov, V. M. Kozintsev, an I. I. Ponomarev, *Residual Stresses in Deformable Solids* [in Russian], Nauka, Moscow (1996).
3. V. P. Myasnikov and M. A. Guzev, "Geometrical model of internal self-equilibrated stresses in solids," *Dokl. Ross. Akad. Nauk*, **38**, No. 5, 627–629 (2001).
4. L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, Pergamon Press, Oxford–New York (1970).
5. K. Kondo, "On the geometrical and physical foundations of the theory of yielding," *Proc. Jpn Nat. Congr. Appl. Mech.*, **2**, 41–47 (1953).
6. B. A. Bilby, R. Bullough, and E. Bullough, "Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry," *Proc. Roy. Soc. London*, **231**, 263–273 (1955).
7. B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern Geometry: Methods and Applications* [in Russian], Nauka, Moscow (1986).
8. L. D. Landau and E. M. Lifshitz, *Field Theory* [in Russian], Nauka, Moscow (1988).
9. S. P. Timoshenko and J. Goodier, *Theory of Elasticity*, McGraw-Hill, New York (1970).
10. I. I. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York (1980).
11. V. Novacki, *Theory of Elasticity* [Russian translation], Mir, Moscow (1975).
12. M. A. Guzev and V. P. Myasnikov, "Thermomechanical model of an elastoplastic material with structural defects," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 4, 156–172 (1998).
13. V. P. Myasnikov and M. A. Guzev, "A geometrical model of the defect structure of an elastoplastic continuous medium," *J. Appl. Mech. Tech. Phys.*, **40**, No. 2, 331–341 (1999).
14. V. P. Myasnikov and M. A. Guzev, "Non-Euclidean model of deformation of materials at different structural levels," *Fiz. Mezomekh.*, **3**, No. 1, 5–16 (2000).